A decision functional for multitime controllability

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Abstract

This paper investigates the multitime linear normal PDE systems. We study especially the controllability of such systems, obtaining complementary results to those in our recent papers. Here the multitime controllability original results are formulated using the γ - gramian matrix, the Im - gramian space and a controllability functional. There are given also some original examples which illustrate and round the theoretical results.

Keywords: multitime controllability, duality, controllability functional, controllability space, Im - gramian space.

1 Introduction

This article studies the controllability of multitime linear PDE systems using new ingredients. Our results in this direction are complementary to those contained in the papers [8], [9], [1] – [3], [6], [7], [20]. The multitime optimal control, and especially the multitime maximum principle, was developed in [4], [5], [10] – [18], [19].

We shall introduce and study new concepts, as for example:

– the *Im - gramian space*; it is a generalization of the controllability gramian image which was defined in the papers [8], [3] only in certain supplementary conditions (the relations (5) from this paper),

– the *controllability γ - functional* and the *controllability functional*; they appear mixing our ideas with those of [20], where a similar single-time functional is presented.

The basic original results of this paper are the Theorems 4 (of Section 4), 5 (of Section 5) and 6 (of Section 6). The Theorem 4 gives necessary conditions of multitime controllability, expressed by the *Im - gramian space*. The Theorem 5 studies necessary conditions of multitime controllability, expressed by the *controllability γ - functional*. The Theorem 6 contains necessary and sufficient conditions of multitime controllability, expressed by the *controllability functional*.

2 Preliminary results

2.1 Controllability of multitime linear PDE systems

Let $D \subseteq \mathbb{R}^m$ be an open and convex subset. We consider the evolution PDE system

$$\frac{\partial x}{\partial t^\alpha} = M_\alpha(t)x + N_\alpha(t)u_\alpha(t), \quad \forall \alpha = \overline{1,m}, \tag{1}$$

where $t = (t^1, \dots, t^m) \in \mathbb{R}^m$, called *multitime*, and $x = (x^1, \dots, x^n)^\top : D \to \mathbb{R}^n = \mathcal{M}_{n,1}(\mathbb{R})$. Also $M_\alpha : D \to \mathcal{M}_n(\mathbb{R})$ are $\mathcal{C}^1$ quadratic matrix functions, $N_\alpha : D \to \mathcal{M}_{n,k}(\mathbb{R})$ are $\mathcal{C}^1$ rectangular matrix functions and $u_\alpha : D \to \mathbb{R}^k = \mathcal{M}_{k,1}(\mathbb{R})$ are $\mathcal{C}^1$ *vector control functions*, all indexed after $\alpha = \overline{1,m}$.

The PDE system (1) is called *completely integrable* if $\forall (t_0, x_0) \in D \times \mathbb{R}^n$, there exists an open set $D_0 \subseteq D \subseteq \mathbb{R}^m$, with $t_0 \in D_0$ and $\exists x : D_0 \to \mathbb{R}^n$, $x(\cdot)$ differentiable, such that $x(\cdot)$ verifies equations (1) on D_0 and $x(t_0) = x_0$. In this case $x(\cdot)$ will be called a *solution* for Cauchy problem $\{(1), x(t_0) = x_0\}$.

The system (1) is completely integrable if and only if the following relations

$$\frac{\partial M_\alpha}{\partial t^\beta} + M_\alpha(t)M_\beta(t) = \frac{\partial M_\beta}{\partial t^\alpha} + M_\beta(t)M_\alpha(t), \tag{2}$$

$$\begin{aligned} &M_\alpha(t)N_\beta(t)u_\beta(t) + \frac{\partial N_\alpha}{\partial t^\beta}u_\alpha(t) + N_\alpha(t)\frac{\partial u_\alpha}{\partial t^\beta} \\ &= M_\beta(t)N_\alpha(t)u_\alpha(t) + \frac{\partial N_\beta}{\partial t^\alpha}u_\beta(t) + N_\beta(t)\frac{\partial u_\beta}{\partial t^\alpha}, \end{aligned} \tag{3}$$

hold, $\forall t \in D$, $\forall \alpha, \beta = \overline{1, m}$. In these conditions, any solution $x(\cdot)$ will be a $\mathcal{C}^2$ function and can be uniquely extended to a global solution ($x : D \to \mathbb{R}^n$). If two solutions coincide at a point, then they will coincide on whole set D.

Thereafter, by a solution we mean a global solution. In the papers [8], [3], it was shown that if the relations (2), (3) are true, then the solution of the Cauchy problem $\{(1),\ x(t_0) = x_0\}$ is

$$x(t) = \chi(t, t_0)x_0 + \int\limits_{\gamma_{t_0,t}} \chi(t, s)N_\alpha(s)u_\alpha(s)\mathrm{d}s^\alpha, \quad \forall t \in D, \tag{4}$$

where $\gamma_{t_0,t}$ is a piecewise $\mathcal{C}^1$ curve included in D, traversed from the multitime t_0 to the multitime t and $\chi(t, s)$ is the fundamental matrix associated to the PDE system, i.e., the matrix solution of the Cauchy problem (see [8])

$$\frac{\partial \chi}{\partial t^\alpha}(t, s) = M_\alpha(t)\chi(t, s), \quad \chi(s, s) = I_n, \quad \forall \alpha = \overline{1, m}.$$

Definition 1. Suppose that the matrix functions $M_\alpha(\cdot)$ verify the relations (2), $\forall t \in D$, $\forall \alpha, \beta = \overline{1, m}$. The vector space

$$\mathcal{U} = \Big\{ u = (u_\alpha)_{\alpha=\overline{1,m}} \ \Big| \ u_\alpha : D \to \mathbb{R}^k = \mathcal{M}_{k,1}(\mathbb{R}), \text{of class } \mathcal{C}^1, \forall \alpha = \overline{1, m}$$

$$\text{and which verify the relations (3) for all } \alpha, \beta \Big\}$$

is called the *control space*, associated to the system (1). If $u \in \mathcal{U}$, we say that u is a *control*.

So, if the matrices $M_\alpha(\cdot)$ verify the relations (2), $\forall t \in D$ and $\forall \alpha, \beta = \overline{1, m}$, then the system (1) is completely integrable if and only if $(u_\alpha)_{\alpha=\overline{1,m}}$ is a control function.

Definition 2. Let us consider the PDE system (1), with the matrix functions $M_\alpha(\cdot)$ verifying the relations (2).

a) The phase (t, x) is called *controllable* if there exists a point $s \in D$, with $s^\alpha > t^\alpha$, $\forall \alpha$, and there exists a control $u(\cdot)$ which transfers the phase (t, x) into the phase $(s, 0)$.

b) Let $t_0, t \in D$, with $t_0^\alpha < t^\alpha$, $\forall \alpha$. The PDE system (1) is called *completely controllable* from t_0 to t if for any point $x \in \mathbb{R}^n$, the phase (t_0, x) transfers into the phase $(t, 0)$, i.e., for any point x the phase (t_0, x) is controllable with the same t.

Let us consider the PDE system (1), with the matrix functions $M_\alpha(\cdot)$ verifying the relations (2). Taking $t_0, t \in D$, we consider the set

$$\mathcal{V}(t_0,t) := \left\{ \int\limits_{\gamma_{t_0,t}} \chi(t_0,s)N_\alpha(s)u_\alpha(s)\mathrm{d}s^\alpha \,\Big|\, (u_\alpha)_{\alpha=\overline{1,m}} \text{ is a control} \right\}.$$

If $(u_\alpha)_{\alpha=\overline{1,m}}$ is a control, then the curvilinear integral

$$\int\limits_{\gamma_{t_0,t}} \chi(t_0,s)N_\alpha(s)u_\alpha(s)\mathrm{d}s^\alpha$$

is path independent, so $\mathcal{V}(t_0,t)$ does not depend on the curve $\gamma_{t_0,t}$, which joins t_0 to t, but it depends on the multitimes t_0 and t.

One remarks immediately that the set $\mathcal{V}(t_0,t)$ is a vector subspace of $\mathbb{R}^n$.

Definition 3. The space $\mathcal{V}(t_0,t)$ is called the *controllability space.*

Theorem 1. *Let us consider the PDE system* (1)*, with the matrix functions $M_\alpha(\cdot)$ verifying the relations* (2)*.*

i) The control $(u_\alpha)_{\alpha=\overline{1,m}}$ transfers the phase (t_0,x_0) into the phase (t,y) if and only if $\chi(t_0,t)y - x_0 = \int\limits_{\gamma_{t_0,t}} \chi(t_0,s)N_\alpha(s)u_\alpha(s)ds^\alpha.$

ii) The phase (t_0,x_0) transfers into the phase (t,y) if and only if

$$x_0 - \chi(t_0,t)y \in \mathcal{V}(t_0,t).$$

iii) The phase (t_0,x_0) is controllable if and only if $\exists t \in D$, with $t^\alpha > t_0^\alpha$, $\forall\alpha$ such that $x_0 \in \mathcal{V}(t_0,t)$.

iv) Let $t_0, t \in D$, with $t_0^\alpha < t^\alpha$, $\forall\alpha$. The PDE system is completely controllable from the multitime t_0 into the multitime t if and only if $\mathcal{V}(t_0,t) = \mathbb{R}^n$.

Proof. The first statement $i)$ is a consequence of the formula (4) and the properties of the fundamental matrix. The second statement $ii)$ follows from $i)$ and the definition of $\mathcal{V}(t_0,t)$. From $ii)$ (taking $y = 0$) and from the Definition 2, one obtains $iii)$ and $iv)$. □

Proposition 1. (see [8], [3]) *Let us suppose that the matrices $(M_\alpha(\cdot))_{\alpha=\overline{1,m}}$ verify the relations* (2)*, $\forall t \in D$, $\forall \alpha, \beta = \overline{1,m}$. We fix $t_0 \in D$. For each $v \in \mathbb{R}^n$ and $\alpha = \overline{1,m}$, we introduce the functions*

$$u_{\alpha,v} : D \to \mathbb{R}^k, \quad u_{\alpha,v}(s) = N_\alpha^\top(s)\chi(t_0,s)^\top v, \quad \forall s \in D.$$

The following statements are equivalent

i) For any $v \in \mathbb{R}^n$, $(u_{\alpha,v})_{\alpha=\overline{1,m}}$ *is a control for the PDE system* (1).

ii) For any $\alpha, \beta = \overline{1,m}$, *the following relations are satisfied on the set* D*:*

$$\begin{aligned} & M_\alpha N_\beta N_\beta^\top + \frac{\partial N_\alpha}{\partial s^\beta} N_\alpha^\top + N_\alpha \frac{\partial N_\alpha^\top}{\partial s^\beta} + N_\beta N_\beta^\top M_\alpha^\top \\ = & M_\beta N_\alpha N_\alpha^\top + \frac{\partial N_\beta}{\partial s^\alpha} N_\beta^\top + N_\beta \frac{\partial N_\beta^\top}{\partial s^\alpha} + N_\alpha N_\alpha^\top M_\beta^\top . \end{aligned} \tag{5}$$

iii) The curvilinear integral $\displaystyle\int_\gamma \chi(t_0, s) N_\alpha(s) N_\alpha^\top(s) \chi(t_0, s)^\top \, ds^\alpha$ *is path independent on* D.

2.2 Quadratic affine forms on Hilbert spaces

In this Section we shall reformulate two well-known Theorems regarding the quadratic affine forms on a Hilbert space. These Theorems will be used in the next Sections to obtain new results concerning the controllability of multitime PDE systems.

Theorem 2. *Let* $\mathcal{H}$ *be a real Hilbert space and* $T : \mathcal{H} \to \mathcal{H}$ *be a linear, continuous, self-adjoint, positive semidefinite operator. For each* $w \in \mathcal{H}$*, we consider the quadratic affine form*

$$F_w : \mathcal{H} \to \mathbb{R}, \quad F_w(v) = \langle T(v), v \rangle - 2\langle w, v \rangle, \quad \forall v \in \mathcal{H}.$$

i) If $v_0 \in \mathcal{H}$ *is a local minimum point of* F_w*, then* $T(v_0) = w$.

ii) If $v_0 \in \mathcal{H}$ *satisfies* $T(v_0) = w$*, then* v_0 *is a global minimum point of* F_w.

iii) If there exists a local maximum point $v_0 \in \mathcal{H}$ *for* F_w*, then* $w = 0$ *and* $T(v) = 0$, $\forall v \in \mathcal{H}$*, i.e., the function* F_w *is identically zero (and, evidently, in this case, any point is a global minimum point).*

iv) v_0 *is a local extremum point of* F_w *if and only if* v_0 *is a global minimum point for* F_w.

Theorem 3. *One consider* $\mathcal{H}$ *as a real Hilbert space of finite dimension* n*. Let* $T : \mathcal{H} \to \mathcal{H}$ *be an self-adjoint, positive semidefinite, linear operator. Let* A *be the matrix of* T*, associated in an arbitrary basis.*

For each $w \in \mathcal{H}$*, we consider the function*

$$F_w : \mathcal{H} \to \mathbb{R}, \quad F_w(v) = \langle T(v), v \rangle - 2\langle w, v \rangle, \quad \forall v \in \mathcal{H}.$$

Then the following statements are equivalent:

i) T is positive definite.
ii) T is bijective (equivalent to $\operatorname{rank} A = n$*).*
iii) For any $w \in \mathcal{H}$, the function F_w has at most a minimum point.
iv) For each $w \in \mathcal{H}$, the function F_w has at least a minimum point.
v) For each $w \in \mathcal{H}$, there exists a unique minimum point for F_w.

Remark 1. According to the Theorem 2, it follows that in the Theorem 3, the minimum point can be understood either a local minimum point or a global minimum point, or a local extremum point.

3 Increasing curves and curvilinear integrals

Here we analyse the notion of increasing curve (respectively, decreasing curve) and we prove some propositions concerning the curvilinear integrals.

Lemma 1. *Let $f : [a, b] \to \mathbb{R}$ be a derivable function with the property that for any ξ_1, $\xi_2 \in [a, b]$, with $\xi_1 < \xi_2$, we have $f(\xi_1) < f(\xi_2)$ (i.e., f is strictly increasing). Suppose there exists $c \in [a, b]$, such that $f'(c) = 0$. Then there exists a sequence $\eta_p \in [a, b]$, with $f'(\eta_p) \neq 0$, $\forall p$, such that $\lim\limits_{p\to\infty} \eta_p = c$.*

Definition 4. Let $t_0, t \in \mathbb{R}^m$, such that $t_0^\alpha \leq t^\alpha$ (respectively $t_0^\alpha \geq t^\alpha$), $\forall\alpha = \overline{1, m}$ and let $\gamma : [a, b] \to \mathbb{R}^m$, $\gamma(\tau) = \Big(\gamma^\alpha(\tau)\Big)_{\alpha=\overline{1,m}}$, with $\gamma(a) = t_0$, $\gamma(b) = t$, a piecewise $\mathcal{C}^1$ curve. We say that γ *increases* (respectively *decreases*) from t_0 to t, if for any $\tau_1, \tau_2 \in [a, b]$, with $\tau_1 < \tau_2$, we have

$$\begin{cases} \gamma^\alpha(\tau_1) < \gamma^\alpha(\tau_2) \,\text{(respectively } \gamma^\alpha(\tau_1) > \gamma^\alpha(\tau_2)), & \text{if} \quad t_0^\alpha \neq t^\alpha; \\ \gamma^\alpha(\tau_1) = \gamma^\alpha(\tau_2), & \text{if} \quad t_0^\alpha = t^\alpha. \end{cases}$$

Remark 2. Let $t_0, t \in D$, such that $t_0^\alpha \leq t^\alpha$ (respectively $t_0^\alpha \geq t^\alpha$), $\forall\alpha = \overline{1, m}$. There exists at least one $\mathcal{C}^1$ curve, included in D, which increases (respectively decreases) from the multitime t_0 to the multitime t. For example, the straight line segment $[t_0, t]$, parameterized by: $\gamma : [0, 1] \to D$, $\gamma(\tau) = (1 - \tau)t_0 + \tau t$, $\forall\tau \in [0, 1]$, is increasing (respectively decreasing) and included in D (since D is a convex set).

Lemma 2. *Let $t_0, t \in D$, such that $t_0^\alpha \leq t^\alpha$ (respectively $t_0^\alpha \geq t^\alpha$), $\forall\alpha = \overline{1, m}$ and let $\gamma : [a, b] \to D$, with $\gamma(a) = t_0$, $\gamma(b) = t$, a piecewise $\mathcal{C}^1$ curve which increases (respectively decreases) from t_0 to t.*

If $P_1, P_2, \ldots, P_m : D \to [0, \infty)$ are continuous functions and $\displaystyle\int_\gamma P_\alpha(s)\, ds^\alpha = 0$, then, for any $\alpha = \overline{1, m}$, with $t^\alpha \neq t_0^\alpha$, and for any $\tau \in [a, b]$, we have $P_\alpha(\gamma(\tau)) = 0$.

Proof. There exist the real numbers $\tau_0, \tau_1, \dots, \tau_q$, such that

$$a = \tau_0 < \tau_1 < \dots < \tau_q = b, \quad \text{with } q \geq 1,\ q \in \mathbb{N},$$

and the curve γ is of class $\mathcal{C}^1$ on each subinterval $[\tau_j, \tau_{j+1}]$, $j = \overline{0, q-1}$.

We know that $\sum_{j=0}^{q-1} \int_{\tau_j}^{\tau_{j+1}} \sum_{\alpha=1}^{m} P_\alpha(\gamma(\tau))\dot{\gamma^\alpha}(\tau)\,\mathrm{d}\tau = 0$. But, for α, with $t^\alpha = t_0^\alpha$, the function $\gamma^\alpha(\cdot)$ is constant on $[\tau_0, \tau_q]$ (see the Definition 4). Consequently, $\dot{\gamma^\alpha}(\tau) = 0$, $\forall \tau \in [\tau_0, \tau_q]$, hence

$$\sum_{j=0}^{q-1} \int_{\tau_j}^{\tau_{j+1}} \sum_{\alpha \text{ with } t^\alpha \neq t_0^\alpha} P_\alpha(\gamma(\tau))\dot{\gamma^\alpha}(\tau)\,\mathrm{d}\tau = 0.$$

Since γ^α is an increasing curve, on each subinterval $[\tau_j, \tau_{j+1}]$, we have $\dot{\gamma^\alpha}(\tau) \geq 0$. We deduce that each from the q integrals, which are terms in the foregoing sum, are ≥ 0. In fact, each integral vanishes as term in a sum equal to zero, i.e.,

$$\int_{\tau_j}^{\tau_{j+1}} \sum_{\alpha \text{ with } t^\alpha \neq t_0^\alpha} P_\alpha(\gamma(\tau))\dot{\gamma^\alpha}(\tau)\,\mathrm{d}\tau = 0, \quad \forall j = \overline{0, q-1}.$$

Since the functions that appear in the sum under the integral are continuous and greater or equal to zero, it follows that for each index $j = \overline{0, q-1}$, and for any α, with $t^\alpha \neq t_0^\alpha$, we have $P_\alpha(\gamma(\tau))\dot{\gamma^\alpha}(\tau) = 0$, $\forall \tau \in [\tau_j, \tau_{j+1}]$. It follows that for each α, with $t^\alpha \neq t_0^\alpha$, we have

$$P_\alpha(\gamma(\tau)) = 0, \quad \forall \tau \in [\tau_j, \tau_{j+1}],\ \text{with } \dot{\gamma^\alpha}(\tau) \neq 0. \tag{$**$}$$

Let α with $t^\alpha \neq t_0^\alpha$, α arbitrarily, but fixed. Let $c \in [\tau_j, \tau_{j+1}]$, with $\dot{\gamma^\alpha}(c) = 0$.

We apply Lemma 1 to the function $\gamma^\alpha(\cdot)$, on the interval $[\tau_j, \tau_{j+1}]$. Hence there exists a sequence $\eta_p \in [\tau_j, \tau_{j+1}]$, with $\dot{\gamma^\alpha}(\eta_p) \neq 0, \forall p$, such that $\lim_{p\to\infty} \eta_p = c$.

Since $\dot{\gamma^\alpha}(\eta_p) \neq 0$, according to $(**)$, we have $P_\alpha(\gamma(\eta_p)) = 0$, $\forall p$. Consequently

$$0 = \lim_{p\to\infty} P_\alpha(\gamma(\eta_p)) = P_\alpha(\gamma(c)).$$

In this way we showed that $P_\alpha(\gamma(c)) = 0$, $\forall c \in [\tau_j, \tau_{j+1}]$, with $\dot{\gamma^\alpha}(c) = 0$. Hence, via $(**)$, it follows $P_\alpha(\gamma(\tau)) = 0$, $\forall \tau \in [\tau_j, \tau_{j+1}]$; and since $j = \overline{0, q-1}$ is arbitrarily, we find $P_\alpha(\gamma(\tau)) = 0$, $\forall \tau \in [a, b]$. $\square$

4 Im - gramian space and conditions for multitime controllability

In the papers [8], [3] it was defined the controllability gramian, $\mathcal{C}(t_0, t)$, but only in the case that the relations (5) hold. In this Section we shall extend the definition to the general situation. But in this case, when the relations (5) are not indispensable true, the gramian will depend on the curve γ. It appears the notion of γ - gramian matrix. Instead of controllability gramian image we shall introduce the Im - gramian space $\mathcal{W}(t_0, t)$. We formulate necessary conditions of controllability expressed using the space $\mathcal{W}(t_0, t)$.

Definition 5. Let us suppose that the matrices $M_\alpha(\cdot)$ verify the relations (2).

i) Let $\gamma : [a, b] \to D$ be a piecewise $\mathcal{C}^1$ curve, with fixed origin $t_0 = \gamma(a)$. The matrix

$$\mathcal{C}_\gamma := \int_\gamma \chi(t_0, s) N_\alpha(s) N_\alpha^\top(s) \chi(t_0, s)^\top \mathrm{d}s^\alpha$$

is called γ - *gramian matrix* associated to the PDE system (1).

ii) Suppose that, for any $\alpha, \beta = \overline{1, m}$, the relations (5) are true. According to the Proposition 1, in this case, the curvilinear integral from *i)* depends only on the ends points and not on the curve joining these ends. Let $t_0, t \in D$ and $\gamma_{t_0,t} : [a, b] \to D$ be a piecewise $\mathcal{C}^1$ curve, with $\gamma_{t_0,t}(a) = t_0$ and $\gamma_{t_0,t}(b) = t$. The matrix

$$\mathcal{C}(t_0, t) := \int_{\gamma_{t_0,t}} \chi(t_0, s) N_\alpha(s) N_\alpha^\top(s) \chi(t_0, s)^\top \mathrm{d}s^\alpha$$

is called the *controllability gramian*.

Remark 3. Let us suppose that the matrices $M_\alpha(\cdot)$ verify the relations (2). Let $\gamma : [a, b] \to D$ be a piecewise $\mathcal{C}^1$ curve, and $t_0 := \gamma(a)$, $t := \gamma(b)$. Let $\gamma^- : [a, b] \to D$, $\gamma^-(\tau) = \gamma(a + b - \tau)$, $\forall \tau \in [a, b]$. Obviously, we have $\gamma^-(a) = t$, $\gamma^-(b) = t_0$.

One verifies immediately

$$\chi(t, t_0) \mathcal{C}_\gamma \chi(t, t_0)^\top = -\mathcal{C}_{\gamma^-}, \quad \text{hence} \quad \operatorname{rank}(\mathcal{C}_\gamma) = \operatorname{rank}(\mathcal{C}_{\gamma^-}).$$

Taking into account that $\chi(t, t_0)^\top$ is invertible, it follows that the equality $\mathrm{Im}(\mathcal{C}_{\gamma^-}) = \chi(t, t_0) \mathrm{Im}(\mathcal{C}_\gamma)$ holds.

Remark 4. Let $t_0, t \in \mathbb{R}^m$, such that $t_0^\alpha \leq t^\alpha$ (respectively $t_0^\alpha \geq t^\alpha$), $\forall \alpha = \overline{1, m}$ and let $\gamma : [a, b] \to \mathbb{R}^m$, with $\gamma(a) = t_0$, $\gamma(b) = t$, a piecewise $\mathcal{C}^1$ curve.

One remarks that γ increases (respectively decreases) from t_0 to t, if and only if γ^- decreases (respectively increases) from t_0 to t.

Definition 6. Suppose that the matrices $M_\alpha(\cdot)$ verify the relations (2). Let $t_0, t \in D$, such that $t_0^\alpha \leq t^\alpha$ (respectively $t_0^\alpha \geq t^\alpha$), $\forall \alpha = \overline{1, m}$. We consider the set

$$\mathcal{W}(t_0, t) := \bigcap_{\gamma_{t_0,t}} \mathrm{Im}(\mathcal{C}_{\gamma_{t_0,t}}) \tag{6}$$

where the intersection is taken over all curves $\gamma_{t_0,t}$, of piecewise $\mathcal{C}^1$ class, included in D, which increases (respectively decreases) from t_0 to t. The set $\mathcal{W}(t_0, t)$ is a vector subspace of $\mathbb{R}^n$ and we shall call it the *Im - gramian space.*

Remark 5. From the Remarks 3 and 4, it follows that, in the conditions of Definition 6, we have

$$\mathcal{W}(t, t_0) = \chi(t, t_0)\mathcal{W}(t_0, t). \tag{7}$$

Remark 6. Furthermore, if for any $\alpha, \beta = \overline{1, m}$, the relations (5) are true, then $\mathcal{W}(t_0, t) = \mathrm{Im}(\mathcal{C}(t_0, t))$ (see also the Definition 5).

Proposition 2. *Suppose that the matrices $M_\alpha(\cdot)$ verify the relations* (2)*. Let $t_0, t \in D$, such that $t_0^\alpha \leq t^\alpha$ (respectively, $t_0^\alpha \geq t^\alpha$), $\forall \alpha = \overline{1, m}$. Then*

$$\mathcal{V}(t_0, t) \subseteq \mathcal{W}(t_0, t). \tag{8}$$

Proof. Let γ be a piecewise $\mathcal{C}^1$ curve, included in D, which increases (respectively, decreases) from t_0 to t. It is sufficient to prove the inclusion

$$\mathcal{V}(t_0, t) \subseteq \mathrm{Im}(\mathcal{C}_\gamma), \tag{9}$$

which is equivalent to $(\mathcal{V}(t_0, t))^\perp \supseteq (\mathrm{Im}(\mathcal{C}_\gamma))^\perp = \mathrm{Ker}((\mathcal{C}_\gamma)^\top)$.

We have $v \in \mathrm{Ker}((\mathcal{C}_\gamma)^\top) \iff ((\mathcal{C}_\gamma)^\top)v = 0 \iff v^\top \mathcal{C}_\gamma = 0$. Hence $v^\top \mathcal{C}_\gamma v = 0$, equivalent to

$$\int_\gamma v^\top \chi(t_0, s) N_\alpha(s) N_\alpha^\top(s) \chi(t_0, s)^\top v \, \mathrm{d}s^\alpha = \int_\gamma \left\| v^\top \chi(t_0, s) N_\alpha(s) \right\|^2 \mathrm{d}s^\alpha = 0.$$

We apply Lemma 2, for $P_\alpha(s) = \left\| v^\top \chi(t_0, s) N_\alpha(s) \right\|^2$ and it follows that for any α, with $t^\alpha \neq t_0^\alpha$, and for any τ, we have $v^\top \chi(t_0, \gamma(\tau)) N_\alpha(\gamma(\tau)) = 0$.

Let now $(u_\alpha(\cdot))_{\alpha=\overline{1,m}}$ be an arbitrary control. We get

$$\left\langle \int\limits_\gamma \chi(t_0,s)N_\alpha(s)u_\alpha(s)\,\mathrm{d}s^\alpha\,;\,v \right\rangle = v^\top \int\limits_\gamma \chi(t_0,s)N_\alpha(s)u_\alpha(s)\,\mathrm{d}s^\alpha$$

$$= \int\limits_\gamma \sum_{\alpha \text{ with } t^\alpha \neq t_0^\alpha} v^\top \chi(t_0,s)N_\alpha(s)u_\alpha(s)\,\mathrm{d}s^\alpha = 0.$$

Hence $v \in (\mathcal{V}(t_0,t))^\perp$, whence we obtain the inclusion

$$\mathrm{Ker}((\mathcal{C}_\gamma)^\top) \subseteq (\mathcal{V}(t_0,t))^\perp.$$

$\square$

From the Theorem 1 and Proposition 2 one obtains immediately the next Theorem (in which we give necessary conditions for controllability):

Theorem 4. *Let us consider the PDE system* (1), *with the matrix functions* $M_\alpha(\cdot)$ *verifying the relations* (2).

i) Let $t_0, t \in D$, *such that* $t_0^\alpha \le t^\alpha$ *(respectively,* $t_0^\alpha \ge t^\alpha$*),* $\forall \alpha = \overline{1,m}$. *If the phase* (t_0, x_0) *transfers into the phase* (t, y), *then*

$$x_0 - \chi(t_0,t)y \in \mathcal{W}(t_0,t).$$

ii) If the phase (t_0, x_0) *is controllable, then* $\exists t \in D$, *with* $t^\alpha > t_0^\alpha$, $\forall \alpha$, *such that* $x_0 \in \mathcal{W}(t_0,t)$.

iii) Let $t_0, t \in D$, *with* $t_0^\alpha < t^\alpha$, $\forall \alpha$. *If the PDE system* (1) *is completely controllable from the multitime* t_0 *into the multitime* t, *then* $\mathcal{W}(t_0,t) = \mathbb{R}^n$.

Remark 7. If the hypotheses of Proposition 2 are completed by the relations (5), then the inclusion (9) becomes equality ([3]), i.e., $\mathcal{V}(t_0,t) = \mathrm{Im}(\mathcal{C}(t_0,t)) = \mathcal{W}(t_0,t)$. In these conditions, one deduces a theorem, similar to the Theorem 4, but in which one obtains necessary and sufficient conditions for controllability.

Generally, the inclusion (8) is strictly. We shall justify this statement by the following example:

Example 1. We consider: $m = 2$, $n = 2$, $k = 2$,

$$M_1 = M_2 = 0, \qquad N_2 = 0, \qquad N_1(s^1,s^2) = \begin{pmatrix} s^2 & 0 \\ 0 & s^2 \end{pmatrix} = s^2 I_2,$$

$$a)\; D = \mathbb{R}^2; \qquad b)\; D = \left\{ \left(t^1,t^2\right) \in \mathbb{R}^2 \,\middle|\, t^2 > 0 \right\} = \mathbb{R} \times (0,\infty).$$

We shall show that for any $t_0, t \in D$, with $t_0^1 < t^1$, $t_0^2 < t^2$, in the case $a)$, we have $\mathcal{V}(t_0,t) = 0$ and $\mathcal{W}(t_0,t) = \mathbb{R}^2 = \mathcal{M}_{2,1}(\mathbb{R})$, while in the case $b)$, the equalities $\mathcal{V}(t_0,t) = \mathcal{W}(t_0,t) = \mathbb{R}^2$ hold.

We have $\chi(s_0, s) = I_2$, $\forall (s_0, s) \in \mathbb{R}^2 \times \mathbb{R}^2$.

a) According to the Definition 1, the pair (u_1, u_2) is a control if and only if (3) is true, i.e.,

$$\frac{\partial}{\partial s^2}\left(N_1(s^1, s^2)\right) u_1(s^1, s^2) + N_1(s^1, s^2)\frac{\partial}{\partial s^2}\left(u_1(s^1, s^2)\right) = 0, \quad \forall (s^1, s^2) \in \mathbb{R}^2,$$

or

$$\frac{\partial}{\partial s^2}\left(s^2 u_1(s^1, s^2)\right) = 0, \quad \forall (s^1, s^2) \in \mathbb{R}^2, \tag{10}$$

equivalent to the fact that there exists a $\mathcal{C}^1$ function $f : \mathbb{R} \to \mathbb{R}^2 = \mathcal{M}_{2,1}(\mathbb{R})$, such that

$$s^2 u_1(s^1, s^2) = f(s^1), \quad \forall (s^1, s^2) \in \mathbb{R}^2.$$

Setting $s^2 = 0$, one obtains $f(s^1) = 0$, $\forall s^1 \in \mathbb{R}$. Hence

$$s^2 u_1(s^1, s^2) = 0, \quad \forall (s^1, s^2) \in \mathbb{R}^2,$$

whence it follows $u_1(s^1, s^2) = 0$, $\forall (s^1, s^2) \in \mathbb{R}^2$, with $s^2 \neq 0$. From the continuity of u_1, we deduce that

$$u_1(s^1, s^2) = 0, \quad \forall (s^1, s^2) \in \mathbb{R}^2.$$

For $u_1 = 0$ and for any u_2, the relation (10) is verified.

Hence the set of controls is made of the pairs (u_1, u_2), with $u_1 = 0$ and $u_2 : \mathbb{R}^2 \to \mathbb{R}^2 = \mathcal{M}_{2,1}(\mathbb{R})$ as an arbitrary $\mathcal{C}^1$ function.

Let γ be a piecewise $\mathcal{C}^1$ curve which increases from the multitime t_0 to the multitime t. Since u_1 and N_2 vanish, it follows

$$\int_\gamma \chi(t_0, s) N_1(s) u_1(s) \mathrm{d}s^1 + \chi(t_0, s) N_2(s) u_2(s) \mathrm{d}s^2 = 0,$$

hence $\mathcal{V}(t_0, t) = 0$ (here it does no matter that γ is increasing, since the curvilinear integral is path independent).

According to the definition, $\mathcal{C}_\gamma = \int_\gamma (s^2)^2 I_2 \mathrm{d}s^1 = \left(\int_\gamma (s^2)^2 \mathrm{d}s^1\right) I_2$. We show that $\int_\gamma (s^2)^2 \mathrm{d}s^1 \neq 0$. Indeed, suppose that we would have $\int_\gamma (s^2)^2 \mathrm{d}s^1 = 0$. According to Lemma 2, it follows that for any τ, we have $(\gamma^2(\tau))^2 = 0$, hence γ^2 is constant (zero), whence we deduce that $t_0^2 = t^2$, which is false.

We deduce that the matrix $\mathcal{C}_\gamma$ has the rank 2. Hence $\mathrm{Im}(\mathcal{C}_\gamma) = \mathbb{R}^2$; it follows the equality $\mathcal{W}(t_0, t) = \mathbb{R}^2$.

b) In the same manner as in the case *a*), one deduces that (u_1, u_2) is a control if and only if the relation (10) is true on the set D. Let $v \in \mathbb{R}^2 = \mathcal{M}_{2,1}(\mathbb{R})$. We select

$$u_1(s^1, s^2) = \frac{1}{s^2(t^1 - t_0^1)} \cdot v, \quad u_2(s^1, s^2) = 0, \quad \forall (s^1, s^2) \in \mathbb{R} \times (0, \infty) = D.$$

We remark immediately that for any pair (u_1, u_2), the relations (10) is true, hence (u_1, u_2) is a control.

Let γ be a piecewise $\mathcal{C}^1$ curve, included in D, which joins t_0 to t. Let us determine $\mathcal{V}(t_0, t)$. One can easily obtain

$$\int_\gamma \chi(t_0, s) N_1(s) u_1(s) \mathrm{d}s^1 + \chi(t_0, s) N_2(s) u_2(s) \mathrm{d}s^2 = v.$$

Since v is arbitrary, it follows $\mathcal{V}(t_0, t) = \mathbb{R}^2$, and from (8) it follows also the equality $\mathcal{W}(t_0, t) = \mathbb{R}^2$.

Remark 8. For the system of Example 1, case *a*), $D = \mathbb{R}^2$, the inclusion (8) is strictly. Also, to the same extent, one deduces that the converse of the Theorem 4 is not always true. Indeed, for any $t_0, t \in \mathbb{R}^2$, with $t_0^1 < t^1$, $t_0^2 < t^2$, we have $\mathcal{V}(t_0, t) = 0$, hence from the Theorem 1, *iii*), it follows that no state (t_0, x_0), with $x_0 \neq 0$, is controllable; however $\mathcal{W}(t_0, t) = \mathbb{R}^2 = \mathbb{R}^n$.

For the same system, considered in the case *b*), $D = \mathbb{R} \times (0, \infty)$, for any $t_0, t \in D$, with $t_0^1 < t^1$, $t_0^2 < t^2$, we have $\mathcal{V}(t_0, t) = \mathbb{R}^2 = \mathbb{R}^n$, whence according to the Theorem 1, *iv*), it follows that the system is completely controllable.

One remarks that for the controllability, it is important the domain on which we consider the matrix functions $M_\alpha(\cdot)$ and $N_\alpha(\cdot)$ which define the system.

5 A decision functional for multitime controllability

Roughly speaking there are essentially two types of methods to study the controllability of linear PDE, namely direct methods and dual methods.

Suppose that the matrices $M_\alpha(\cdot)$ verify the relations (2). We consider the adjoint (dual) PDE system of the PDE system (1), i.e.,

$$\frac{\partial \varphi}{\partial t^\alpha} = -(M_\alpha(t))^\top \varphi, \quad \forall \alpha = \overline{1, m} \tag{11}$$

Let $t_0 \in D$ and $v \in \mathbb{R}^n$. The Cauchy problem $\{(11), \varphi(t_0) = v\}$ has the solution

$$\varphi_v(\,\cdot\,, t_0) : D \to \mathbb{R}^n, \quad \varphi_v(s, t_0) = \chi(t_0, s)^\top v, \quad \forall s \in D. \tag{12}$$

Denote by $\mathcal{S}$, the set of solutions of the PDE system (11), i.e.,

$$\mathcal{S} = \left\{ \varphi : D \to \mathbb{R}^n \;\middle|\; \varphi \text{ solution of the system (11)} \right\} = \left\{ \varphi_v(\,\cdot\,, t_0) \;\middle|\; v \in \mathbb{R}^n \right\}.$$

First, we remark that $\mathcal{S}$ is a real vector space.

Let $x_0 \in \mathbb{R}^n$, $t_0, t \in D$, and $\gamma_{t_0,t}$ be a piecewise $\mathcal{C}^1$ curve, included in D, which joins the multitime t_0 to the multitime t (covered from t_0 to t). We define the functional $F_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t) : \mathcal{S} \to \mathbb{R}$,

$$F_{\gamma_{t_0,t}}(\varphi, x_0; t_0, t) = \int\limits_{\gamma_{t_0,t}} \left\| N_\alpha(s)^\top \varphi(s) \right\|^2 \mathrm{d}s^\alpha - 2\langle x_0, \varphi(t_0)\rangle, \quad \forall \varphi \in \mathcal{S}, \tag{13}$$

and we call it the *controllability $\gamma_{t_0,t}$ - functional.*

The function $L_{t_0} : \mathbb{R}^n \to \mathcal{S}$, $L_{t_0}(v) = \varphi_v(\,\cdot\,, t_0)$ is obvious an isomorphism of vector spaces. Its inverse is $L_{t_0}^{-1} : \mathcal{S} \to \mathbb{R}^n$, $L_{t_0}^{-1}(\varphi) = \varphi(t_0)$, $\forall \varphi \in \mathcal{S}$.

Now define the function

$$\widetilde{F}_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t) : \mathbb{R}^n \to \mathbb{R}, \;\; \widetilde{F}_{\gamma_{t_0,t}}(v, x_0; t_0, t) = F_{\gamma_{t_0,t}}(L_{t_0}(v), x_0; t_0, t). \tag{14}$$

Let us write $\widetilde{F}_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$ by a formula showing that it is in fact a quadratic affine form. In this sense, we have

$$\widetilde{F}_{\gamma_{t_0,t}}(v, x_0; t_0, t) = \left\langle \mathcal{C}_{\gamma_{t_0,t}} v\,, v \right\rangle - 2\left\langle x_0, v \right\rangle, \quad \forall v \in \mathbb{R}^n. \tag{15}$$

Remark 9. From (14) and from the fact that L_{t_0} is a bijective function, we deduce that v is an extremum point (respectively minimum, maximum) for $\widetilde{F}_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$ if and only if $L_{t_0}(v) = \varphi_v(\,\cdot\,, t_0)$ is an extremum point (respectively minimum, maximum) for $F_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$. And conversely: φ is an extremum point (respectively minimum, maximum) for $F_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$ if and only if $L_{t_0}^{-1}(\varphi) = \varphi(t_0)$ is an extremum point (respectively minimum, maximum) for $\widetilde{F}_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$. Hence L_{t_0} is a bijection between the extremum point set (respectively minimum, maximum) of $\widetilde{F}_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$ and the extremum point set (respectively minimum, maximum) of $F_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$. The notions of extremum (respectively minimum, maximum) are understood globally, for the time being.

In order to speak of local extremum points, we shall give a topology on $\mathcal{S}$. Since L_{t_0} is an isomorphism of vector spaces, it follows that $\mathcal{S}$ has

finite dimension, namely n. We endow $\mathcal{S}$ with the topology induced by an arbitrary norm. There exists a norm on $\mathcal{S}$, for example, $\|\varphi\|_{\mathcal{S}} = \|L_{t_0}^{-1}(\varphi)\| = \|\varphi(t_0)\|$. Since $\mathcal{S}$ has finite dimension, any two norms are equivalent, hence they induce the same topology. Now, L_{t_0} is a homeomorphism between $\mathbb{R}^n$ and $\mathcal{S}$. We deduce that L_{t_0} is a bijection between the set of local extremum points (respectively local minimum, local maxim) of $\widetilde{F}_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$ and the set of local extremum points (respectively local minimum, local maxim) of $F_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$.

Throughout, in the sequel, we suppose that the matrices $M_\alpha(\cdot)$ verify the relations (2), on the set D.

Let $t_0, t \in D$, with $t_0^\alpha \leq t^\alpha$, $\forall \alpha$ and let $\gamma_{t_0,t} : [a, b] \to D$ be a piecewise $\mathcal{C}^1$ curve, which increase from t_0 to t. Then

$$\left\langle \mathcal{C}_{\gamma_{t_0,t}} v \,, v \right\rangle = \int\limits_{\gamma_{t_0,t}} \left\| N_\alpha(s)^\top \chi(t_0, s)^\top v \right\|^2 \mathrm{d}s^\alpha \geq 0, \quad \forall v \in \mathbb{R}^n, \tag{16}$$

The inequality (16) holds since $\dot{\gamma}_{t_0,t}^\alpha(\tau) \geq 0$, $\forall \tau$ (eventually, for a finite number of points, we have lateral derivative).

The matrix $\mathcal{C}_{\gamma_{t_0,t}}$ is symmetric. From now on and from (16), we deduce that the function $\widetilde{F}_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$ takes the form of the functional F_w, of Theorems 2 and 3. Here, $\mathcal{H} = \mathbb{R}^n$, $\dim \mathcal{H} = n < \infty$, $T(v) = \mathcal{C}_{\gamma_{t_0,t}} v$; T is linear, hence also continuous, since $\mathcal{H}$ has finite dimension; T is self-adjoint and positive semidefinite (of (16)). Also $w = x_0$.

Consequently, we can apply the Theorems 2 and 3, for $\widetilde{F}_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$.

It follows that the local extremum points of $\widetilde{F}_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$, if they exist, are in fact global minimum points. According to the Remark 9, it follows that the same thing happen also for $F_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$. Therefore, in the sequel we shall refer only to (global) minimum points.

From the Theorem 2 and the Remark 9, we find immediately

Proposition 3. *Let $t_0, t \in D$, with $t_0^\alpha \leq t^\alpha$, $\forall \alpha$, let $\gamma_{t_0,t} : [a, b] \to D$ be a piecewise $\mathcal{C}^1$ curve, which increases from t_0 to t, and let $x_0 \in \mathbb{R}^n$. Then the following statements are equivalent:*

i) $F_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$ *has at least one minimum point.*

ii) $\widetilde{F}_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$ *has at least a minimum point.*

iii) $x_0 \in \mathrm{Im}(\mathcal{C}_{\gamma_{t_0,t}})$.

From the Theorem 3 and the Remark 9, it follows immediately

Proposition 4. *Let $t_0, t \in D$, with $t_0^\alpha \leq t^\alpha$, $\forall \alpha$ and let $\gamma_{t_0,t} : [a,b] \to D$ be a piecewise $\mathcal{C}^1$ curve, increasing from t_0 to t. Then the following statements are equivalent:*

i) For any $x_0 \in \mathbb{R}^n$, there exists a unique minimum point for the functional $F_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$.

ii) For any $x_0 \in \mathbb{R}^n$, there exists a unique minimum point for the functional $\widetilde{F}_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$.

iii) $\operatorname{rank}(\mathcal{C}_{\gamma_{t_0,t}}) = n$.

From the Propositions 3, 4 and Theorem 4, it follows directly:

Theorem 5. *Let us consider the PDE system* (1), *with the matrix functions $M_\alpha(\cdot)$ verifying the relations* (2).

i) Let $t_0, t \in D$, such that $t_0^\alpha \leq t^\alpha$, $\forall \alpha = \overline{1,m}$. If the phase (t_0, x_0) transfers into the phase (t, y), then for any piecewise $\mathcal{C}^1$ curve $\gamma_{t_0,t}$, included in D, increasing from t_0 to t, it follows that $F_{\gamma_{t_0,t}}(\,\cdot\,, x_0 - \chi(t_0,t)y; t_0, t)$ and $\widetilde{F}_{\gamma_{t_0,t}}(\,\cdot\,, x_0 - \chi(t_0,t)y; t_0, t)$ have global minimum points.

ii) If the phase (t_0, x_0) is controllable, then $\exists t \in D$, with $t^\alpha > t_0^\alpha$, $\forall \alpha$, such that for any piecewise $\mathcal{C}^1$ curve $\gamma_{t_0,t}$, included in D, increasing from t_0 to t, it follows that $F_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$ and $\widetilde{F}_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$ have global minimum points.

iii) Let $t_0, t \in D$, with $t_0^\alpha < t^\alpha$, $\forall \alpha$. If the PDE system (1) *is completely controllable from the multitime t_0 into the multitime t, then for any piecewise $\mathcal{C}^1$ curve $\gamma_{t_0,t}$, included in D, increasing from t_0 to t, and for any $x_0 \in \mathbb{R}^n$, there exists a unique (global) minimum point for $F_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$.*

Analogously for $\widetilde{F}_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$.

6 Controllability functional

We consider the PDE system (1), for which the matrix functions $M_\alpha(\cdot)$ and $N_\alpha(\cdot)$ verify the relations (2) and (5), on the set D.

Let $x_0 \in \mathbb{R}^n$ and $t_0, t \in D$. Let $\gamma_{t_0,t}$ be a piecewise $\mathcal{C}^1$ curve, included in D. According to (15), we have

$$\widetilde{F}_{\gamma_{t_0,t}}(v, x_0; t_0, t) = \left\langle \mathcal{C}_{\gamma_{t_0,t}} v \,, v \right\rangle - 2 \left\langle x_0, v \right\rangle, \quad \forall v \in \mathbb{R}^n.$$

But we have seen (Proposition 1, Definition 5) that in this case, the functional $\mathcal{C}_{\gamma_{t_0,t}}$ does not depend on the curve $\gamma_{t_0,t}$, but only on the ends t_0, t. It will be denoted by $\mathcal{C}(t_0, t)$. From here and from (15), it follows that also

$\widetilde{F}_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$ does not depend on the curve $\gamma_{t_0,t}$, but only on the ends t_0, t. We denote

$$\widetilde{F}(v, x_0; t_0, t) := \widetilde{F}_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t) = \langle \mathcal{C}(t_0, t)v\,, v\rangle - 2\,\langle x_0, v\rangle\,, \ \ \forall v \in \mathbb{R}^n. \tag{17}$$

Since (14)

$$F_{\gamma_{t_0,t}}(\varphi, x_0; t_0, t) = \widetilde{F}_{\gamma_{t_0,t}}(L_{t_0}^{-1}(\varphi), x_0; t_0, t) = \widetilde{F}(L_{t_0}^{-1}(\varphi), x_0; t_0, t), \quad \forall \varphi \in \mathcal{S},$$

it follows that $F_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$ depends only on the ends t_0, t, and not on the curve $\gamma_{t_0,t}$. We denote

$$F(\,\cdot\,, x_0; t_0, t) := F_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t),$$

and we call it the *controllability functional.*

Let $t_0, t \in D$, with $t_0^\alpha < t^\alpha$, $\forall\alpha$. From the Theorem 1 and Remark 7, it follows that if the relations (5) hold, then the PDE system (1) is completely controllable from t_0 to t if and only if $\operatorname{rank}\mathcal{C}(t_0, t) = n$; from the Theorem 2, Theorem 3, Remark 9, we find immediately:

Theorem 6. *Let us consider the PDE system* (1)*, with the matrix functions* $M_\alpha(\cdot)$*,* $N_\alpha(\cdot)$ *verifying the relations* (2) *and* (5)*, on* D*.*

i) Let $t_0, t \in D$*,* $t_0 \neq t$*,* $t_0^\alpha \leq t^\alpha$*,* $\forall\alpha$*. Then the phase* (t_0, x_0) *transfers to the phase* (t, y) *if and only if* $F(\,\cdot\,, x_0 - \chi(t_0, t)y; t_0, t)$ *has at least a minimum point.*

ii) The phase (t_0, x_0) *is controllable if and only if* $\exists t \in D$*, with* $t^\alpha > t_0^\alpha$*,* $\forall\alpha$*, such that* $F(\,\cdot\,, x_0; t_0, t)$ *has at least a minimum point.*

iii) Let $t_0, t \in D$*, with* $t_0^\alpha < t^\alpha$*,* $\forall\alpha$*. The PDE system* (1) *is completely controllable from* t_0 *to* t *if and only if for any* $x_0 \in \mathbb{R}^n$*, there exists a unique global minimum point, for the functional* $F(\,\cdot\,, x_0; t_0, t)$*.*

iv) Let $t_0, t \in D$*, with* $t_0^\alpha < t^\alpha$*,* $\forall\alpha$*. The PDE system* (1) *is completely controllable from* t_0 *to* t *if and only if for any* $x_0 \in \mathbb{R}^n$*, the functional* $F(\,\cdot\,, x_0; t_0, t)$ *has at least a minimum point.*

v) Let $t_0, t \in D$*, with* $t_0^\alpha < t^\alpha$*,* $\forall\alpha$*. The PDE system* (1) *is completely controllable from* t_0 *to* t *if and only if for any* $x_0 \in \mathbb{R}^n$*, the functional* $F(\,\cdot\,, x_0; t_0, t)$ *has at most a minimum point.*

Analogously for $\widetilde{F}(\,\cdot\,, x_0; t_0, t)$*.*

7 An unbounded extention of controllability functional

Let us give an example of a complete controllable PDE system, with the matrices M_α, N_α, verifying the relations (2) and (5). The Theorem 5 says

that for t_0, t, with $t_0^\alpha < t^\alpha$, and for any curve $\gamma_{t_0,t}$, which increases from t_0 to t, the controllability $\gamma_{t_0,t}$ - functional, has a unique global minimum point on the set $\mathcal{S}$.

The Formula (13), which defines the functional $F_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$, has sense also on some spaces $\mathcal{S}_1$, which contains $\mathcal{S}$, as would be the space $\mathcal{C}^r(D;\mathbb{R}^n)$ with $r \in \{0,1,2\}$. This means that we can introduce a new functional

$$J_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t) : \mathcal{S}_1 \to \mathbb{R},$$

$$J_{\gamma_{t_0,t}}(\varphi, x_0; t_0, t) = \int\limits_{\gamma_{t_0,t}} \left\| N_\alpha(s)^\top \varphi(s) \right\|^2 \mathrm{d}s^\alpha - 2\langle x_0, \varphi(t_0)\rangle, \quad \forall \varphi \in \mathcal{S}_1. \tag{18}$$

The functional $J_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$ is obviously an extension of $F_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$.

We select $\mathcal{S}_1 = \mathcal{C}^2(D;\mathbb{R}^n)$. In the case of PDE system in our example, we shall show that for any $x_0 \neq 0$, for any t_0, t, with $t_0^\alpha < t^\alpha$, $\forall \alpha$, and for any curve $\gamma_{t_0,t}$, which increases from t_0 to t, the extention $J_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$ is unbounded from below. Hence the Theorems 5 and 6 are no longer valid on spaces including strictly the set $\mathcal{S}$.

Now, suppose $m = 2,\ n = 2,\ k = 1,\ D = \mathbb{R}^2$,

$$M_1(s) = M_2(s) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ N_1(s) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ N_2(s) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \forall s \in \mathbb{R}^2.$$

The relations (2) and (5) hold. We have $\chi(s_1, s_2) = I_2$, $\forall s_1, s_2 \in \mathbb{R}^2$.

Let $t_0 = (t_0^1, t_0^2)$, $t = (t^1, t^2)$, with $t_0^1 < t^1$, $t_0^2 < t^2$ and $x_0 = (a, b)^\top$.

For $v = (v_1, v_2)^\top \in \mathbb{R}^2 = \mathcal{M}_{2,1}(\mathbb{R})$, we find $\varphi_v(s, t_0) = \chi(t_0, s)^\top v = v$, $\forall s \in \mathbb{R}^2$;

$$N_1^\top(s)\varphi_v(s, t_0) = v_1, \quad N_2^\top(s)\varphi_v(s, t_0) = v_2, \quad \forall s \in \mathbb{R}^2.$$

$$F(\,\cdot\,, x_0; t_0, t) : \mathcal{S} \to \mathbb{R}, \quad F\Big(\varphi_v(\,\cdot\,, t_0), x_0; t_0, t\Big) =$$

$$= (t^1 - t_0^1)\cdot\left(v_1 - \frac{a}{t^1 - t_0^1}\right)^2 + (t^2 - t_0^2)\cdot\left(v_2 - \frac{b}{t^2 - t_0^2}\right)^2$$

$$-\frac{a^2}{t^1 - t_0^1} - \frac{b^2}{t^2 - t_0^2} \geq -\frac{a^2}{t^1 - t_0^1} - \frac{b^2}{t^2 - t_0^2},$$

with equality if and only if $v_1 = \dfrac{a}{t^1 - t_0^1}$ and $v_2 = \dfrac{b}{t^2 - t_0^2}$.

Hence, for any $t_0 = (t_0^1, t_0^2)$, $t = (t^1, t^2)$, with $t_0^1 < t^1$, $t_0^2 < t^2$ and any $x_0 = (a, b)^\top$, the functional $F(\,\cdot\,, x_0; t_0, t)$ has a unique global minimum point, which is $\varphi_{v_0}(\,\cdot\,, t_0)$, where $v_0 = \left(\dfrac{a}{t^1 - t_0^1}, \dfrac{b}{t^2 - t_0^2}\right)$.

From the Theorem 6, it follows that the system is completeley controllable.

We denote $\mathcal{S}_1 = \left\{ \varphi : \mathbb{R}^2 = D \to \mathbb{R}^2 \;\middle|\; \varphi \text{ of class } \mathcal{C}^2 \right\}$. Obviously $\mathcal{S} \subseteq \mathcal{S}_1$.

Let $\gamma_{t_0,t}$ be an increasing curve from t_0 to t. We select $x_0 = (a, b)^\top \neq (0, 0)^\top$. Define $J_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$, by the formula (18). We obtain an extension of the controllability functional to the space $\mathcal{S}_1$.

Let $q > 0$. If $a + b \neq 0$, we select $c = a + b$. If $b = -a$, we select $c = a$; we have $c \neq 0$ since $x_0 \neq 0$. We choose

$$\varphi_1(s^1, s^2) = c \cdot \sqrt{\frac{q}{1 + q^2(s^1 + s^2 - t_0^1 - t_0^2)^2}}, \quad \forall (s^1, s^2) \in \mathbb{R}^2.$$

If $a + b \neq 0$, we consider the function $\varphi_2(\cdot) = \varphi_1(\cdot)$. If $b = -a$, we take the function $\varphi_2(\cdot) = -\varphi_1(\cdot)$. We select $\varphi(\cdot) = (\varphi_1(\cdot), \varphi_2(\cdot))^\top$. Obviously, $\varphi \in \mathcal{S}_1$.

In the case $a + b \neq 0$, we obtain

$$J_{\gamma_{t_0,t}}(\varphi, x_0; t_0, t) = (a + b)^2 \arctan\Big(q(t^1 + t^2 - t_0^1 - t_0^2)\Big) - 2(a + b)^2\sqrt{q}.$$

In case $b = -a \neq 0$, we find

$$J_{\gamma_{t_0,t}}(\varphi, x_0; t_0, t) = a^2 \arctan\Big(q(t^1 + t^2 - t_0^1 - t_0^2)\Big) - 4a^2\sqrt{q}.$$

In both cases we have $\lim\limits_{q\to\infty} J_{\gamma_{t_0,t}}(\varphi, x_0; t_0, t) = -\infty$.

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